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Solution of Newell-Whitehead-Segel equation with power law nonlinearity and time dependent coefficients

Research Article

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Abstract: The present paper is connected with obtaining the exact solution of the generalized Newell-Whitehead-Segel(gNWS) equation with power law nonlinearity and with time dependent coefficients using tanh function method. It is necessary for the time dependent coefficients to be Riemann integrable.

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Keywords: Newell-Whitehead-segel equation • Power law nonlinearity • tanh function method

1. Introduction

The nonlinear evolution equations(NLEEs) play an important and significant role in modelling various physical phenomena related to solid state physics,fluid mechanics, plasma physics, population dynamics, chemical kinetics, nonlinear optics, protein chemistry etc.Newell-Whitehead equation is a NLEE which was originated in the famous paper of Newell and Whitehead [1] for Be′nard's problem. Be′nard's problem is a hydrodynamic problem in which water contained between two plates is heated from below. It exibits patterns like rolls, hexagons or rectangles if the bifurcation parameter which is related to temperature differences between the plates is above a certain threshhold. The equation derived by Newell-Whitehead-Segel(NWS) [1, 2] is of the form

$$u_t(x,t) - au_{xx}(x,t) = bu(x,t) - cu^3(x,t)$$

where a is a (+)ve interger and b and c are real numbers. Over the years researchers have applied various methods to solve the NWS equation. Malomed B.[8] proposed dispersive NWS equation for travelling wave pattern in binary fluids. Manaa [3] applied Adomain Decomposition method and multiquadratic quasi-interpolation method. Aasaari A. [4] and Sarvanan et. al. [5] used differential transform method and Hassan [6] used Homotopy Perturbation method to NWS equation. Malik et.al.[7] applied $\left(\frac{G'}{G}\right)$ expansion method to obtain generalized travelling wave solutions of NWS equation.

Of late, NLEEs with time dependent /periodic coefficients [9–13] have attracted the attention of many reserarchers. In this paper, we have discussed the generalized NWS (gNWS) equation with power law nonlinearity in the presence of linear damping and diffusion, and having time dependent coefficients, given by

$$u_t + \alpha(t)u^n u_x + \beta(t)u_{xx} = \gamma(t)u(1-u^n) \tag{1}$$

for (+)ve integer n. Here u_t describes the time dependent evolution term , $u^n u_x$ and u^{n+1} are the terms with power law nonlinearity, u_{xx} is the diffusion term. u represents linear damping. $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are the Riemann integrable time dependent coefficients. Due to the presence of time dependent coefficients, the gNWS equation can not be integrated by classical integration methods. In this paper, gNWS Eq. (1) is effectively integrated by tanh function method.

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2. Tanh function method

The beauty of tanh function is that all the derivatives of tanh are expressible in terms of tanh. Suppose that a NLEE is given by

$$Q(v, v_t, v_x, v_{xx}, v_{xt}, v_{tt}, ...) = 0 \tag{2}$$

Where $v = v(x,t)$ is unknown function, Q is a polynomial in its arguments. Introducing a new variable

$$Y = tanh(\mu\xi), \quad \xi = p(t)x + q(t) \tag{3}$$

where $p(t)$, $q(t)$ are integrable functions of t to be determined, the PDE (2) is transformed into an ODE of wave variable $\xi = p(t)x + q(t)$ as

$$P(v, v', v'', ...) = 0 \tag{4}$$

where $' = d/d\xi$

Equation (4) is integrated considering all the constants of integration of the derivative terms to be zero. Then the change of derivatives are as follows:

$$\frac{d}{d\xi} = (1-Y^2)\frac{d}{dY}$$

$$\frac{d^2}{d\xi^2} = -2\mu^2 Y(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)\frac{d^2}{dY^2}$$

tanh function method permits the finite expansion

$$v(\mu\xi) = F(Y) = \sum_{i=0}^{m} a_i Y^i + \sum_{i=1}^{m} b_i Y^{-i} \tag{5}$$

for some (+)ve integer m, which in most of the cases is determined by homogeneous balancing technique [14]. The constants a_i and b_i are unknown to be determined. Substituting (5) into (4), the ODE (4) yields an algebraic equation in power of Y. Vanishing all these coefficients then obtained a system of algebraic equations involving the parameters a_k, b_k $(k = 0,1,2,......m)$, μ, $p(t)$, $q(t)$.

3. Solution of Newell-Whitehead-Segel equation by tanh method

In this section we shall investigate the gNWS Eq. Equation 1 with the help of tanh function method described in section 2. A Special case where $\gamma(t) = 0$ and $n = 1$ is the Burger equation while $\gamma(t) = 0$ and $n = 2$ gives rise to the generalized Burger equation.

Sustitute the relation

$$u(x,t) = [\lambda(t)v(\xi)]^{\frac{1}{n}}, \quad \xi = p(t)x + q(t) \tag{6}$$

to transform the Eq. (1) to

$$\begin{aligned}&\lambda(t)v^2\lambda'(t) + \lambda(t)^2 v v'(xp'(t) + q'(t)) + \alpha(t)\lambda(t)^3 v^2 v' p(t) + \beta(t)\Big((\frac{1}{n}-1)\lambda(t)^2 v'^2 p(t)^2 + \lambda(t)^2 v v'' p(t)^2\Big)\\ &= n\gamma(t)(\lambda(t)^2 v^2 - \lambda(t)^3 v^3)\end{aligned} \tag{7}$$

Balancing vv'' and v^3 we get
$m + m + 2 = 3m$ so that
$m = 2$
and we have

$$v(\xi) = a_0 + a_1 Y(\xi) + a_2 Y(\xi)^2 + b_1 Y(\xi)^{-1} + b_2 Y(\xi)^{-2} \tag{8}$$

Substitute (8) in Eq. (7), collect the coefficients of power of $x^j Y^i$, $-7 \le i \le 7$, $j = 0,1$ and set all the coefficients to zero. Y^7 and Y^{-7} yeilds the following system of equations :

$$2a_2^3\mu\alpha(t)\lambda(t)^3 = 0$$
$$2b_2^3\mu\alpha(t)\lambda(t)^3 = 0$$

This results $a_2 = 0$ and $b_2 = 0$ for $\alpha(t) \neq 0$ and $\lambda(t) \neq 0$ The simplified remaining equations are

$$xY^3 : \frac{dp(t)}{dt} = 0 \Longrightarrow p(t) = constants(say \quad p(t) = k)$$
$$Y^4 : -a_1^3\mu\alpha(t)\lambda(t)^3 + a_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 + 2a_1^2\mu^2\beta(t)\lambda(t)^2 = 0$$
$$Y^{-4} : -b_1^3\mu\alpha(t)\lambda(t)^3 + b_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 + 2b_1^2\mu^2\beta(t)\lambda(t)^2 = 0$$
$$Y^3 : -2a_0a_1^2\mu\alpha(t)\lambda(t)^3 + 2a_0a_1\mu^2\beta(t)\lambda(t)^2 + a_1^3n\gamma(t)\lambda(t)^3 - a_1^2\mu\lambda(t)^2q'(t) = 0$$
$$Y^{-3} - 2a_0b_1^2\mu\alpha(t)\lambda(t)^3 + 2a_0b_1\mu^2\beta(t)\lambda(t)^2 + b_1^3n\gamma(t)\lambda(t)^3 - b_1^2\mu\lambda(t)^2q'(t) = 0$$
$$Y^2 : -a_0^2a_1\mu\alpha(t)\lambda(t)^3 + 3a_0a_1^2n\gamma(t)\lambda(t)^3 - a_0a_1\mu\lambda(t)^2q'(t) + a_1^3\mu\alpha(t)\lambda(t)^3$$
$$- a_1^2b_1\mu\alpha(t)\lambda(t)^3 - 2a_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 - a_1^2n\gamma(t)\lambda(t)^2 + a_1^2\lambda(t)\lambda'(t)$$
$$-2a_1^2\mu^2\beta(t)\lambda(t)^2 - 2a_1b_1\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 + 2a_1b_1\mu^2\beta(t)\lambda(t)^2 = 0$$
$$Y^{-2} : -a_0^2b_1\mu\alpha(t)\lambda(t)^3 + 3a_0b_1^2n\gamma(t)\lambda(t)^3 - a_0b_1\mu\lambda(t)^2q'(t) - a_1b_1^2\mu\alpha(t)\lambda(t)^3$$
$$-2a_1b_1\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 + 2a_1b_1\mu^2\beta(t)\lambda(t)^2 + b_1^3\mu\alpha(t)\lambda(t)^3$$
$$-2b_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 - b_1^2n\gamma(t)\lambda(t)^2 + b_1^2\lambda(t)\lambda'(t) - 2b_1^2\mu^2\beta(t)\lambda(t)^2 = 0$$
$$Y : 3a_0^2a_1n\gamma(t)\lambda(t)^3 + 2a_0a_1^2\mu\alpha(t)\lambda(t)^3 - 2a_0a_1n\gamma(t)\lambda(t)^2 + 2a_0a_1\lambda(t)\lambda'(t)$$
$$-2a_0a_1\mu^2\beta(t)\lambda(t)^2 + 3a_1^2b_1n\gamma(t)\lambda(t)^3 + a_1^2\mu\lambda(t)^2q'(t) = 0$$
$$Y^{-1} : 3a_0^2b_1n\gamma(t)\lambda(t)^3 + 2a_0b_1^2\mu\alpha(t)\lambda(t)^3 - 2a_0b_1n\gamma(t)\lambda(t)^2 + 2a_0b_1\lambda(t)\lambda'(t)$$
$$-2a_0b_1\mu^2\beta(t)\lambda(t)^2 + 3a_1b_1^2n\gamma(t)\lambda(t)^3 + b_1^2\mu\lambda(t)^2q'(t) = 0$$
$$Y^0 : a_0^3n\gamma(t)\lambda(t)^3 + a_0^2a_1\mu\alpha(t)\lambda(t)^3 + a_0^2b_1\mu\alpha(t)\lambda(t)^3 - a_0^2n\gamma(t)\lambda(t)^2$$
$$+ a_0^2\lambda(t)\lambda'(t) + 6a_0a_1b_1n\gamma(t)\lambda(t)^3 + a_0a_1\mu\lambda(t)^2q'(t) + a_0b_1\mu\lambda(t)^2q'(t) + a_1^2b_1\mu\alpha(t)\lambda(t)^3$$
$$+ a_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 + a_1b_1^2\mu\alpha(t)\lambda(t)^3 + 4a_1b_1\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2$$
$$-2a_1b_1n\gamma(t)\lambda(t)^2 + 2a_1b_1\lambda(t)\lambda'(t) - 4a_1b_1\mu^2\beta(t)\lambda(t)^2 + b_1^2\mu^2(\frac{1}{n}-1)\beta(t)\lambda(t)^2 = 0$$

Solving thses equations we obtain the following cases with feasible values of the parameters.

Case 1:

$$\begin{aligned} & b_1 = 0, \quad n\alpha(t)\lambda(t) \neq 0, \quad a_1 = \frac{k\mu(n+1)\beta(t)}{n\alpha(t)\lambda(t)}, \quad a_0 \neq 0, \quad a_0 = \frac{\pm\mu(n+1)}{n} or \\ & a_0 = \frac{n^2(n+1)^2\gamma(t) \pm (n^2+n)\sqrt{n^2(n+1)^2\gamma(t)^2 - 4(n-1)^2\mu^2\alpha(t)^2}}{2n^2(n-1)\alpha(t))}, \quad n > 1 \\ & \lambda(t) = \frac{2a_0}{3a_0^2 + a_1^2 - e^{2a_0\int_1^t(-\frac{n\gamma(t)}{2a_0})dt}}, \quad q(t) = \int_1^t \frac{-2a_0a_1k\mu\alpha(t)\lambda(t) + 2a_0k^2\mu^2\beta(t) + a_1^2n\gamma(t)\lambda(t)}{a_1\mu}dt \end{aligned} \tag{9}$$

Case 2:

$$\begin{aligned} & a_1 = 0, \quad n\alpha(t)\lambda(t) \neq 0, \quad b_1 = \frac{k\mu(n+1)\beta(t)}{n\alpha(t)\lambda(t)}, \quad a_0 \neq 0, \quad b_1\mu \neq 0, \quad a_0 = \frac{\pm\mu(n+1)}{n} \quad or \\ & a_0 = \frac{n^2(n+1)^2\gamma(t) \pm (n^2+n)\sqrt{n^2(n+1)^2\gamma(t)^2 - 4(n-1)^2\mu^2\alpha(t)^2}}{2n^2(n-1)\alpha(t))}, \quad n > 1 \\ & \lambda(t) = \frac{2a_0}{3a_0^2 + b_1^2 - e^{2a_0\int_1^t(-\frac{n\gamma(t)}{2a_0})dt}}, \quad q(t) = \int_1^t \frac{-2a_0b_1k\mu\alpha(t)\lambda(t) + 2a_0k^2\mu^2\beta(t) + b_1^2n\gamma(t)\lambda(t)}{b_1\mu}dt \end{aligned} \tag{10}$$

Case 3:

$$\begin{aligned}
&n\alpha(t)\lambda(t) \neq 0, \quad b_1 = \frac{k\mu(n+1)\beta(t)}{n\alpha(t)\lambda(t)}, \quad a_1 = b_1, \quad a_0 \neq 0, \quad \mu(n+1) \neq 0, \quad \beta(t)\gamma(t) \neq 0,\\
&a_0 = \pm\frac{2\mu(1+n)}{n}\\
&or \quad a_0 = \frac{n^2(n+1)^2\gamma(t) \pm (n^2+n)\sqrt{n^2(n+1)^2\gamma(t)^2 - 16(n-1)^2\mu^2\alpha(t)^2}}{2n^2(n-1)\alpha(t))}, \quad n>1\\
&\lambda(t) = \frac{2a_0}{3a_0^2 + 4b_1^2 - e^{2a_0\int_1^t(-\frac{n\gamma(t)}{2a_0})dt}}, \quad q(t) = \int_1^t \frac{-2a_0k\mu\alpha(t)\lambda(t) + b_1(n^2+n)\gamma(t)\lambda(t)}{\mu(n+1)}dt
\end{aligned} \tag{11}$$

Since a_1 , b_1 and a_0 are constants therefore the parameters $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ must satisfy the constraint $\beta(t) = \alpha(t)\lambda(t)$ and $\alpha(t) = \gamma(t)$ in all the above three cases.

Case 4:

This case yields generalized Burger equation. The parameters bear the values as follows:

$$\gamma(t) = 0, \quad n\alpha(t)\lambda(t) \neq 0, \quad b_1 = \frac{k\mu(n+1)\beta(t)}{n\alpha(t)\lambda(t)}, \quad a_1 = b_1, \quad a_0 \neq 0, \quad \lambda(t)\lambda'(t) = 0 \tag{12}$$

Since a_1 and b_1 are constants therefore the parameters $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ must satisfy the constraint $\beta(t) = \alpha(t)\gamma(t)$. This results $a_0 = \pm\frac{2\mu(n+1)}{n}$, $n \neq 0$

$q(t) = -\int_1^t \frac{2k(a_0b_1 - a_0)\mu\alpha(t)\lambda(t)}{b_1}dt$ Therefore we obtain the following solutions for different cases:

The solution for Case 1 is

$$\begin{aligned}
&u(x,t) = [(\frac{2a_0}{3a_0^2 + a_1^2 - e^{-\int_1^t n\gamma(t)dt}})a_0 + a_1 tanh(\xi)]^{\frac{1}{n}} \quad \xi = x + \int_1^t \frac{(-2a_0a_1k\mu + 2a_0k^2\mu^2 + a_1^2n)\alpha(t)\lambda(t)}{a_1\mu}dt\\
&a_0 = \frac{\pm\mu(n+1)}{n}, \quad n \neq 0 \quad or \quad a_0 = \frac{n^2(n+1)^2 \pm (n^2+n)\sqrt{n^2(n+1)^2 - 4(n-1)^2\mu^2}}{2n^2(n-1)}, \quad n>1\\
&a_1 = \frac{k\mu(n+1)}{n}
\end{aligned}$$

The solution for Case 2 is

$$\begin{aligned}
&u(x,t) = [(\frac{2a_0}{3a_0^2 + b_1^2 - e^{-\int_1^t n\gamma(t)dt}})a_0 + b_1 coth(\xi)]^{\frac{1}{n}} \quad \xi = x + \int_1^t \frac{(-2a_0b_1k\mu + 2a_0k^2\mu^2 + b_1^2n)\alpha(t)\lambda(t)}{b_1\mu}dt\\
&a_0 = \frac{\pm\mu(n+1)}{n}, \quad n \neq 0 \quad or \quad a_0 = \frac{n^2(n+1)^2 \pm (n^2+n)\sqrt{n^2(n+1)^2 - 4(n-1)^2\mu^2}}{2n^2(n-1)}, \quad n>1\\
&b_1 = \frac{k\mu(n+1)}{n}
\end{aligned}$$

The solution for Case 3 is

$$\begin{aligned}
&u(x,t) = [\frac{2a_0}{3a_0^2 + 4b_1^2 - e^{-\int_1^t n\gamma(t)dt}}(a_0 + b_1(tanh(\xi) + coth(\xi))]^{\frac{1}{n}},\\
&\xi = x + \int_1^t \frac{-2a_0k\mu\alpha(t)\lambda(t) + b_1(n^2+n)\gamma(t)\lambda(t)}{\mu(n+1)}dt\\
&a_0 = \pm\frac{2\mu(1+n)}{n}, \quad n \neq 0 \quad or \quad a_0 = \frac{n^2(n+1)^2 \pm (n^2+n)\sqrt{n^2(n+1)^2 - 16(n-1)^2\mu^2}}{2n^2(n-1)}, \quad n>1\\
&a_1 = b_1 = \frac{k\mu(n+1)}{n}
\end{aligned}$$

4. Observations

The graphical observations of the solution of the gNWS equation shows that for $n = 1$ the variable coefficients gNWS equations gives kink type solutions for $b_1 = 0$. For non-zero a_0, a_1 andb_1 the solutions are unpredictable for smaller values of x. But as x increases the solutions gradually become stable.

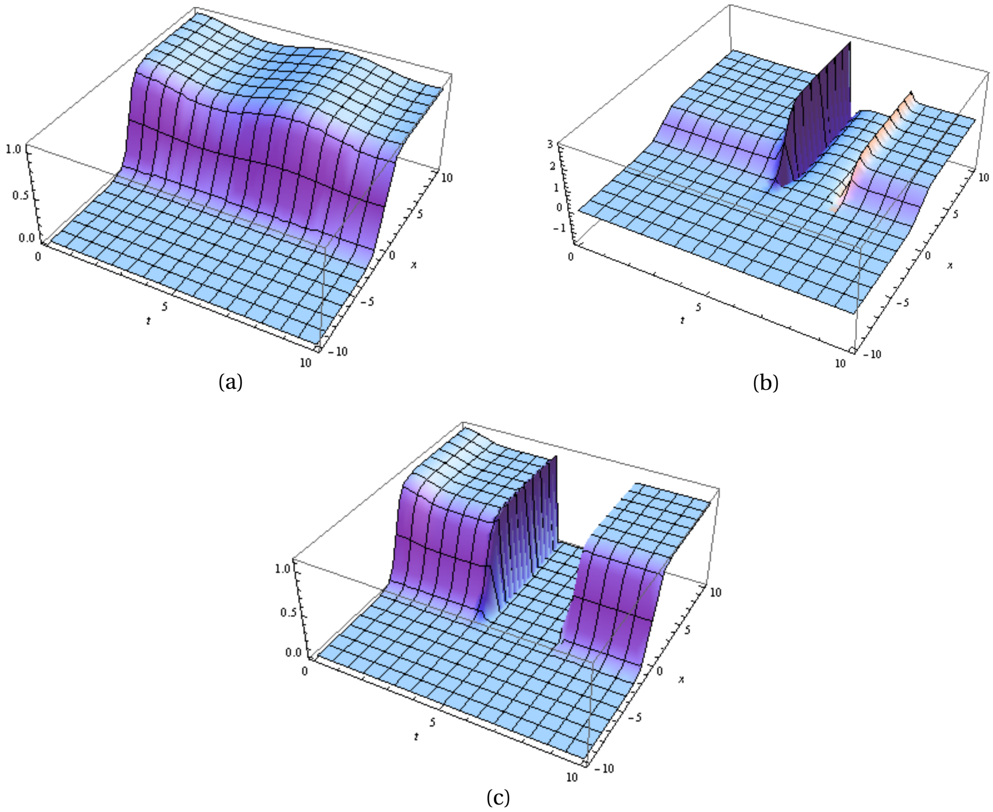

Fig. 1. (a) $\alpha(t) = tan(t)$, (b) $\alpha(t) = tsin(t)$, (c) $\alpha(t) = e^{t} sint$ with $b_1 = 0$ and $n = 1$

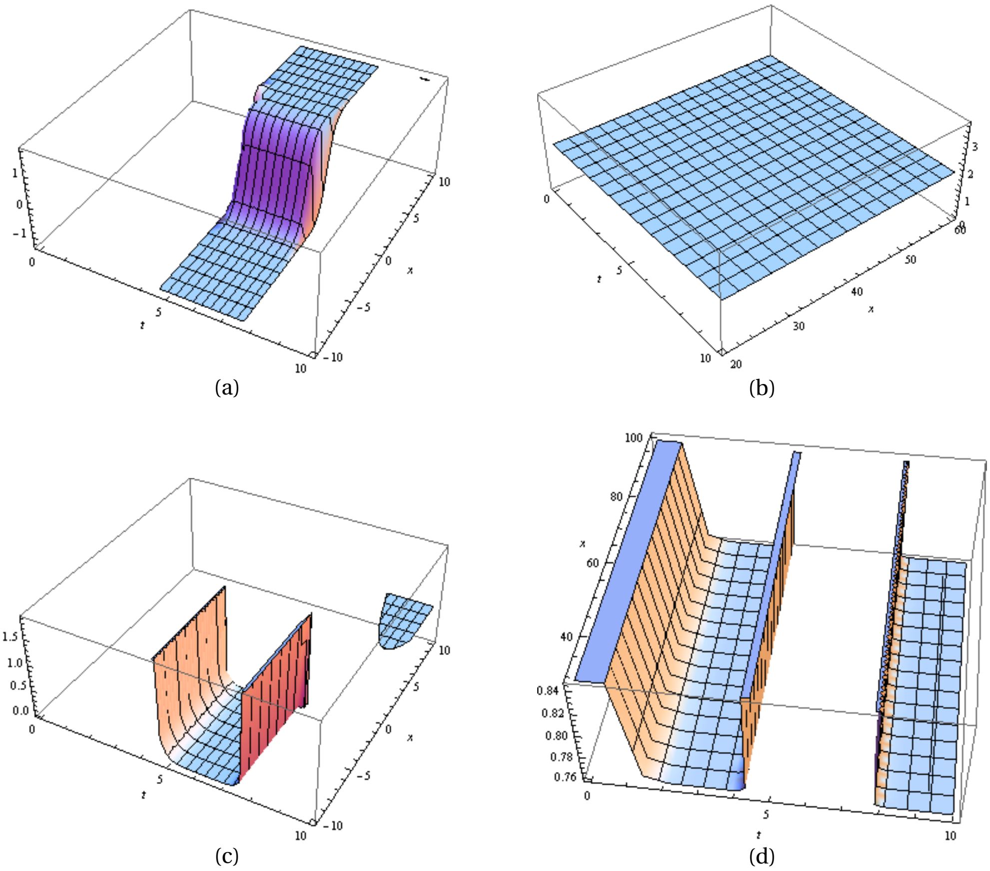

Fig. 2. (a)-(b) $\alpha(t) = tsin(t)$ for n=2, (c)-(d) $\alpha(t) = tsin(t)$ for n=3

5. Conclusion

In this paper, we have shown that tanh function method is an efficient tool for solving gNWS equation. Solutions of gNWS equation are obtained in three different forms de- pending on the values of the unknowns.The generalized Burger equation, as a special case, gets solved naturally. Also we have observed that the time dependent coefficients must satisfy some constraint conditions to retain the originality of the gNWS equation. These constraint conditions in fact help in simplifying the values of the unknowns.

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