Int. J. Adv. Appl. Math. and Mech. 3(2) (2015) 1 – 4 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

A characterization of the symmetric Banzhaf value for Games with a Coalition Structure

Research Article

Jony Rojas-Rojas*, **Francisco Sanchez-Sanchez**

Department of Basic Mathematics, Centro de Investigación en Matemáticas, A.C., Jalisco S/N, 36240, Guanajuato, México

Received 23 September 2015; accepted (in revised version) 09 November 2015

Abstract: We provide a new characterization of the symmetric Banzhaf value via the coalitional neutral player, intracoalitional balanced contributions and coalitional transfer invariance axioms. The latter axiom says that a union's total payoff is invariant to a transfer of worth between two coalitions he or she belongs to, provided that such coalitions do not contain subsets of any union.

MSC: 91A12 • 91A40

Keywords: Games • Value • Coalition Structure

1. Introduction

The Shapley value [1] and the Banzhaf value [2] are the most popular values in games with transferable utilities (or TU-games). Both values assign to each player the sum of weighted marginal contributions. In the Banzhaf value each player has the same probability to enter any coalition and on the Shapley value this probability depends on the number of elements of a coalition. These two values have been extended to TU-games with a coalition structure [3, 4] or a coalition configuration [5, 6], TU-games on antimatroids [7, 8], and TU-games on convex geometries [9, 10], etc.

Alonso-Meijide and Fiestras-Janeiro [11] proposed another extension of the Banzhaf value to TU-games with a coalition structure, called the symmetric Banzhaf value. This value is the composition of the Banzhaf value and the Shapley value in the following sense: suppose that in a TU-game with a union system there are two levels of negotiation; first, between unions and then between players of the same union. The distribution between unions is done by assigning to each union their Banzhaf value in the quotient game. The amount assigned to each union in this first share is divided between the players of each union using the Shapley value of each union's TU-game.

We provide a new and concise characterization of the extension given by Alonso-Meijide and Fiestras-Janeiro in [11] of the Banzhaf value for TU-games with a coalition structure by means of three axioms. The first one is the well-known intracoalitional balanced contribution axiom. The second axiom, called coalitional neutral player, indicates that if a union is a neutral player in the quotient game, then the payoff to that union is what its members obtain on their own in the TU-game. The third axiom, called coalitional transfer invariance, says that a union's total payoff is invariant to a transfer of worth between two coalitions he or she belongs to, provided that such coalitions do not contain subsets of any union. This means that if the worth of a coalition varies a certain amount and, at the same time, the worth of another coalition varies by the opposite amount, then for unions whose members belong to both coalitions the total payoff allocation does not change, provided that such coalitions do not contain subsets of any union. The result so achieved is therefore placeable into the crossing zone between Operations research and management science through a suitable application of decision theory. Due to this feature, the treatment pursued in this work concerns applied mathematics and, for the arguments and disciplines herein involved, it has been kindly accepted by the new IJAMM which has already published many papers drawn up upon the same cross-disciplinary area in which

* Corresponding author.
E-mail addresses: rojasr@cimat.mx (Jony Rojas-Rojas), sanfco@cimat.mx (Francisco Sanchez-Sanchez)

Operations research and management science fruitfully intertwine amongst them. See for instance from [12], [13], [14] and [15].

The rest of this paper is organized as follows. section 2 contains the notation and basic definitions and introduces the axiom of coalitional transfer invariance and coalitional neutral player. Equation 3 is devoted to the proof of the main result of the article.

2. Preliminaries

2.1. TU-game

The cardinality of a finite set T will be denoted by a lowercase letter t and the power set of T by 2^T. For simplicity of notation, we write i instead of $\{i\}$. A *cooperative game with transferable utilities (or simply, TU-game)* is a pair (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. We denote by G^N the set of TU-games on N. For any pair of TU-games $v, w \in G^N$ and any $\alpha \in \mathbb{R}$, the TU-game $\alpha v + w$ is defined as: $(\alpha v + w)(S) = \alpha v(S) + w(S)$, for all $S \subseteq N$. For each non-empty coalition $T \subseteq N$, the Dirac TU-game $\delta_T \in G^N$ is defined as: $\delta_T(S) = 1$ if $S = T$ and $\delta_T(S) = 0$ for any other $S \subseteq N$. A player $i \in N$ is a *neutral player* in (N, v) if $\forall S \ni i$, $v(S) - v(S \setminus i) = v(i)$.

A *value* ϕ in G^N is a mapping $\phi : G^N \to \mathbb{R}^n$ which uniquely determines, for all $(N, v) \in G^N$ and all $i \in N$, a payoff $\phi_i(N, v) \in \mathbb{R}$ for participating in this TU-game $(N, v) \in G^N$.

Dummy player axiom (DP) *A value ϕ satisfies DP if, for each $i \in N$ such that i is a neutral player in (N, v), it holds that: $\phi_i(N, v) = v(i)$.*

Two values in G^N that satisfy DP are the *Shapley value* [1] and the *Banzhaf value* [2]. The Shapley value is the mapping Sh that assigns to each TU-game $(N, v) \in G^N$ and each $i \in N$ the real number $Sh_i(N, v) \in \mathbb{R}$ given by

$$Sh_i(N, v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus i)].$$

The Banzhaf value is the mapping Bz that assigns to each TU-game $(N, v) \in G^N$ and each $i \in N$ the real number $Bz_i(N, v) \in \mathbb{R}$ given by

$$Bz_i(N, v) = \sum_{S \ni i} \frac{1}{2^{n-1}} [v(S) - v(S \setminus i)].$$

For further information about the meaning and properties of the Shapley value and the Banzhaf value, refer to [1] and [16], respectively.

2.2. TU-games with a coalition structure

A *coalition structure* over the set of players, N, is a partition of N, that is, $\mathscr{B} = \{B_1, \ldots, B_m\}$ is a coalition structure if it satisfies that $\bigcup_{1 \le l \le m} B_l = N$ and $B_k \cap B_l = \emptyset$ if $k = l$. We also assume $B_l \neq \emptyset$ for all l. The sets $B_l \in \mathscr{B}$ are called "unions" or "blocks". We denote by $\mathscr{B}(N)$ the set of all coalition structures over N. A TU-game with a coalition structure is a triplet $(N, \mathscr{B}, v)$, where $(N, v) \in G^N$ and $\mathscr{B} \in \mathscr{B}(N)$. The coalition structure $\mathscr{B}$ gives an a priori structure of cooperation among players. We denote by Γ^N the set of TU-games with a coalition structure.

If $(N, \mathscr{B}, v) \in \Gamma^N$ with $\mathscr{B} = \{B_l : l \in M := \{1, \ldots, m\}\} \in \mathscr{B}(N)$, the *quotient game* $(M, v^{\mathscr{B}})$ is a TU-game played by the unions, i.e,

$$(M, v^{\mathscr{B}}) \in G^M \qquad \textit{and} \qquad v^{\mathscr{B}}(K) = v\left(\bigcup_{k \in K} B_k\right) \quad \forall K \subseteq M.$$

A *value* f in Γ^N is a mapping $f : \Gamma^N \to \mathbb{R}^n$ which uniquely determines, for all $(N, \mathscr{B}, v) \in \Gamma^N$ and all $i \in N$, a payoff $f_i(N, \mathscr{B}, v) \in \mathbb{R}$ for participating in this TU-game with a coalition structure $(N, \mathscr{B}, v) \in \Gamma^N$.

For a value f in Γ^N and $S \subseteq N$, let $f(N, \mathscr{B}, v)[S] = \sum_{i \in S} f_i(N, \mathscr{B}, v)$. For all $l \in M$ with $b_l > 1$ and $i \in B_l$, define $\mathscr{B}_{-i} = \{B_1, \ldots, B_l \setminus i, \ldots B_m\}$, that is, $\mathscr{B}_{-i}$ is the new coalition structure when the player i leaves the TU-game.

Our interest is focused on the value proposed in [11], for TU-games with a coalition structure, called the symmetric Banzhaf value. Given a TU-game with a coalition structure $(N, \mathscr{B}, v) \in \Gamma^N$, this value assigns to each player $i \in N$ the real number

$$\Pi_i(N, \mathscr{B}, v) = \sum_{K \ni k} \sum_{\substack{S \ni i \\ S \subseteq B_k}} \frac{1}{2^{m-1}} \frac{(s-1)!(b_k - s)!}{b_k!} [v(Q_K \cup S) - v(Q_K \cup S \setminus i)]$$

where $B_k \in \mathscr{B}$ is the union containing i ($i \in B_k$) and where we set $Q_R = \bigcup_{r \in K} B_r$.

Since $\Pi(N, \mathscr{B}, v)[B_l] = Bz_l(M, v^{\mathscr{B}})$ for all $l \in M$ and the Banzhaf value satisfies DP, then the symmetric Banzhaf value, Π, satisfies the following axiom:

Coalitional neutral player (CNP) *A value f in Γ^N satisfies CNP if, for all $(N,\mathscr{B},v) \in \Gamma^N$ and all $l \in M$ such that l is a neutral player in $(M, v^{\mathscr{B}})$, it holds that: $f(N,\mathscr{B},v)[B_l] = v(B_l)$.*

We note that given $\mathscr{B} \in \mathscr{B}(N)$ and $(N,v),(N,w) \in G^N$ the quotient game of $(N,\mathscr{B},(v+w))$ is $(M,\mathscr{B}, v^{\mathscr{B}} + w^{\mathscr{B}})$. From this linearity, the equation $\Pi(N,\mathscr{B},v)[B_l] = Bz_l(M, v^{\mathscr{B}})$, $\forall l \in M$, the equality $\delta^{\mathscr{B}}_{Q_K}(R) = \delta_K(R)$ for all $R \in 2^M$ and all $K \in 2^M \setminus \emptyset$, plus the fact that $Bz_l(M, v^{\mathscr{B}})$ only depends on the sum of the marginal contributions of coalitions of unions that contain l, we obtain that the symmetric Banzhaf value, Π, satisfies the following axiom:

Coalitional transfer invariance (CTI) *A value f in Γ^N satisfies CTI if, for all $(N,\mathscr{B},v) \in \Gamma^N$, all $\alpha \in \mathbb{R}$ and all $K,R \in 2^M \setminus \emptyset$, it holds that: $f(N,\mathscr{B},v)[B_l] = f(N,\mathscr{B},v+\alpha(\delta_{Q_K} - \delta_{Q_R}))[B_l], \forall l \in K \cap R$.*

3. Results

It turns out that the combination of CNP, CTI and the following axiom characterizes Π on Γ^N.

Intracoalitional balanced contributions (IBC) *A value f in Γ^N satisfies IBC if, for all $(N,\mathscr{B},v)$, for all $l \in M$ and all $\{i,j\} \subseteq B_l$, it holds that:*

$$f_i(N,\mathscr{B},v) - f_i(N \setminus j, \mathscr{B}_{-j}, v) = f_j(N,\mathscr{B},v) - f_j(N \setminus i, \mathscr{B}_{-i}, v).$$

The IBC axiom indicates that if two players $i,j \in N$ belong to the same union, then the contribution of player j to the payoff of i is equal to the contribution of player i to the payoff of j.

The characterization theorem is:

Theorem 3.1.
The only value defined in Γ^N that satisfies CNP, CTI and IBC is the symmetric Banzhaf value.

Proof. **Existence.** In section 2 we saw that the value Π satisfies CNP and CTI. From the results of Myerson [17] and since the value Π is written as

$$\Pi_i(N,\mathscr{B},v) = \frac{1}{2^{m-1}} \sum_{K \ni k} Sh_i(B_k, v_{Q_K}) \quad \forall i \in N,$$

where $v_{Q_K}(S) = v(Q_K \cup S) - v(Q_K)$ for all $S \subseteq B_k$, we deduce that the value Π satisfies IBC.

Uniqueness. Let $(N,\mathscr{B},v) \in \Gamma^N$ be a TU-game with a coalition structure, f a value in Γ^N satisfying the axioms listed in the hypotheses, and $q = (q_S)_{S \in 2^N}$ such that $q_S \in \mathbb{R}$ and $q_\emptyset = 0$. Let us first prove that $f(N,\mathscr{B},v)[B_l] = \Pi(N,\mathscr{B},v)[B_l]$ for all $l \in M$. To do this, we define a value φ^q in G^M such that for all $(M,w) \in G^M$ and all $l \in M$, $\varphi^q_l(M,w) = f(N,\mathscr{B},\overline{w})[B_l]$, where for all $S \subseteq N$, $\overline{w}(S) = w(T)$ if $S = Q_T$ and $\overline{w}(S) = q_S$ otherwise. As f is a value in Γ^N, then φ^q is a value in G^M.

We next prove that φ^q satisfies DP and *strong transfer invariance (STI)* in G^M.

DP. Let $l \in M$ such that l is a neutral player in the TU-game $(M,w) \in G^M$. Given that the quotient game of the TU-game with a coalition structure $(N,\mathscr{B},\overline{w}) \in \Gamma^N$ is precisely $(M,w) \in G^M$ and f satisfies CDP, then $\varphi^q_l(M,w) = f(N,\mathscr{B},\overline{w})[B_l] = \overline{w}(B_l) = w(l)$. This implies that φ^q satisfies DP in G^M.

STI. Let $l \in M$ and (K^+,K^-,α) be a transfer with $l \in K^+ \cap K^-$. Since the TU-game $\overline{w+\alpha(\delta_{K^+} - \delta_{K^-})} \in G^N$ coincides with the TU-game $\overline{w} + \alpha(\delta_{Q_{K^+}} - \delta_{Q_{K^-}}) \in G^N$ we obtain that $\varphi^q_l(M, w+\alpha(\delta_{K^+} - \delta_{K^-})) = f(N,\mathscr{B},\overline{w} + \alpha(\delta_{Q_{K^+}} - \delta_{Q_{K^-}})[B_l]$.

Given that $l \in K^+ \cap K^-$ and f satisfies CTI, then $\varphi^q_l(M, w+\alpha(\delta_{K^+} - \delta_{K^-})) = f(N,\mathscr{B},\overline{w} + \alpha(\delta_{Q_{K^+}} - \delta_{Q_{K^-}})[B_l] = f(N,\mathscr{B},\overline{w})[B_l] = \varphi^q_l(M,w)$, which implies that φ^q satisfies STI in G^M.

Béal et al. [18] proved that the Banzhaf value is the only value in G^M that satisfies DP and STI. Hence, $\varphi^q = Bz$ in G^M.

Consider q such that $q_S = v(S)$ for all $S \in 2^N$. Now, it is trivial to deduce that

$$Bz_l(M, v^{\mathscr{B}}) = f(N,\mathscr{B},v)[B_l] \quad \forall l \in M. \tag{1}$$

From (1) and the fact that $\Pi(N,\mathscr{B},v)[B_l] = Bz_l(M,v^{\mathscr{B}})$, for all $l \in M$, we conclude that

$$f(N,\mathscr{B},v)[B_l] = \Pi(N,\mathscr{B},v)[B_l]. \tag{2}$$

Let $l \in M$. We next prove that $f_i(N,\mathscr{B},v) = \Pi_i(N,\mathscr{B},v)$ for all $i \in B_l$, by induction on b_l. If $B_l = \{j\}$, then the equality (2) implies that $f_j(N,\mathscr{B},v) = \Pi_j(N,\mathscr{B},v)$. Let us assume that $b_l \geq 2$. By IBC for each $\{i,j\} \subseteq B_l$ it holds that

$$f_i(N,\mathscr{B},v) - f_i(N \setminus j, \mathscr{B}_{-j}, v) = f_j(N,\mathscr{B},v) - f_j(N \setminus i, \mathscr{B}_{-i}, v). \tag{3}$$

Using the induction hypothesis, we have $f_i(N \setminus j, \mathscr{B}_{-j}, v) = \Pi_i(N \setminus j, \mathscr{B}_{-j}, v)$ and $f_j(N \setminus i, \mathscr{B}_{-i}, v) = \Pi_j(N \setminus j, \mathscr{B}_{-j}, v)$. By (3) we obtain that $f_i(N,\mathscr{B},v) - f_j(N,\mathscr{B},v) = \Pi_i(N \setminus j, \mathscr{B}_{-j}, v) - \Pi_j(N \setminus i, \mathscr{B}_{-i}, v)$. This implies that $f_i(N,\mathscr{B},v) - f_j(N,\mathscr{B},v) = \Pi_i(N,\mathscr{B},v) - \Pi_j(N,\mathscr{B},v)$. Adding over all $j \in B_l$ in the above equality and using (2) we obtain that $b_l f_i(N,\mathscr{B},v) = b_l \Pi_i(N,\mathscr{B},v)$. From this equality and since $b_l \geq 2$ we conclude that $f_i(N,\mathscr{B},v) = \Pi_i(N,\mathscr{B},v)$ for all $i \in B_l$.

□

Acknowledgments

The authors acknowledge support from CONACyT grants 167924.

References

[1] L. Shapley, A value for n-person games, In Contributions to the Theory of Games Volume II, Annals of Mathematical Studies, Editors H.W. Kuhn and A.W. Tucker. Princeton University Press. 28 (1953) 307–317.
[2] J. F. Banzhaf, Weighted voting doesn't work: A mathematical analysis, Rutgers Law Review. 19 (1965) 317–343.
[3] G. Owen, Essays in Mathematical economics and game theory. R. Henn and O. Moeschlin, New York, 1977.
[4] G. Owen, Modification of the Banzhaf-Coleman index for games with a priori unions, M.J, Holler (Ed.), Power, voting, and voting power, Physica-Verlag, Wurzburg. (1981) 232–238.
[5] M. J. Albizuri, J. Aurrecoechea, J. M. Zarzuelo, Configuration values: Extensions of the coalitional Owen value, Games econ behav. 57 (2006) 1–17.
[6] M. J. Albizuri, J. Aurrecoechea, Coalition configurations and the Banzhaf index, Soc Choice Welf. 26 (2006) 571–596.
[7] E. Algaba, J. M. Bilbao, R. van den Brink, A. Jiménez-Losada, An axiomatization of the Banzhaf value for cooperative games on antimatroids, Math Method Oper Res. 59 (2004) 147–166.
[8] E. Algaba, J. M. Bilbao, R. van den Brink, A. Jiménez-Losada, Axiomatizations of the Shapley value for cooperative games on antimatroids, Math Method Oper Res. 57 (2003) 49–65.
[9] J. Bilbao, P. Edelman, The Shapley value on convex geometries, Discrete Appl Math. 103 (2000) 33 – 40.
[10] J. Bilbao, A. Jiménez, J. López, The Banzhaf power index on convex geometries, Math Soc Sci. 36 (1998) 157–173.
[11] J. M. Alonso-Meijide, M. G. Fiestras-Janeiro, Modification of the Banzhaf value for games with a coalition structure, Ann of Oper Res. 109 (2002) 213–227.
[12] K. Jeganathan, Finite source Markovian inventory system with bonus service for certain customers, Int. J. of Adv. in Aply. Math. and Mech. 2(3) (2015) 134 - 143.
[13] N. Anbazhagan, B. Vigneshwaran, K. Jeganathan, Stochastic inventory system with two types of services, Int. J. of Adv. in Aply. Math. and Mech. 2(1) (2014) 120 - 127.
[14] P. K. Pandit, Supply-and-demand model involving fuzzy parameters, Int. J. of Adv. in Aply. Math. and Mech. 1(2) (2013) 103 - 115.
[15] R. Sajja1, S. R. Chalamalasetti, A selective survey on multi-objective meta-heuristic methods for optimization of master production scheduling using evolutionary approaches, Int. J. Adv. Appl. Math. and Mech. 1(3) (2014) 109 - 120.
[16] P. Dubey, L. S. Shapley, Mathematical properties of the Banzhaf power index, Math Oper Res. 4 (1979) 99–131.
[17] R. B. Myerson, Conference structures and fair allocation rules, Int J Game Theory. 9 (1980) 169–182.
[18] S. Béal, E. Rémila, P. Solal, Decomposition of the space of TU-games, Strong Transfer Invariance and the Banzhaf value, Oper Res Lett. 43 (2015) 123–125.